Name——————————, ID ——————

 © copyright Ayman Badawi 2012

MTH 530, Abstract Algebra I (graduate) Fall 2012 ,HW number SEVEN (Due: Sunday December 23)

Ayman Badawi

QUESTION 1. (i) Let p be a prime number > 3. We know that Z_p^* under multiplication modulo p is a cyclic group of order p - 1. Let $H = \{a^2 \mid a \in Z_p^*\}$. Prove that H is a subgroup of order $(p-1)/2$. [Hint: you may want to use the concept of group homomorphism].

(ii) Let p and H as above (part (i)). Suppose that $p - 1 \notin H$. Prove that for each $a \in Z_P^*$, we have either $a \in H$ or $p - a \in H$.

(iii) Let D be a group of order $n \geq 2$. Prove that D is a group-isomorphic to a subgroup of S_n.

(iv) Let $n \geq 2$ be a positive integer and F be the set of all non-isomorphic groups of order n. Prove that F is a finite set.

(v) Let F be a group of order p^n for some prime number p and positive integer $n \geq 1$. Prove that F has a subgroup of order p^i for each i where $1 \leq i \leq (n-1)$

(vi) Let F be a finite group. Suppose that $p^n \mid |F|$ for some prime number p and positive integer n. Prove that F has a subgroup of order p^n.

(vii) Let G be a group and H be a cyclic group and F is a group homomorphism from G onto H (i.e., Range(F) = H). Is $F^{-1}(H)$ an Abelian group? Prove or Disprove.

(viii) Let D, K be finite groups such that $K < D$. Assume that K is a sylow p-subgroup of D. Let F be a p-subgroup of D such that $F \subseteq N_D(K)$. Prove that $F \subseteq K$.

(ix) Let F be a group such that $|F| = pq$ for some distinct prime numbers p, q where $p < q$ and p does not divide q - 1. Prove that F is cyclic (i.e., D is a group-isomorphic to Z_{pq}).

(x) Let G be a group of order 105 such that $7 \mid |Z(G)|$. Prove that G is an abelian group.

(xi) Let G be a group of order 56. Prove that H is not a simple group.

(xii) Let G be a group of order 345. Prove that G is an abelian group. Can we say more about G?

(xiii) Let D be a finite simple group and suppose that D has two subgroups, say K and H, such that $[D : K] = p$ and $[D : H] = q$ for some prime numbers p, q. Prove that $|H| = |K|$ (and hence $p = q$). [hint: indeed interesting!!!!]

(xiv) Let F be a simple group of order 60. Prove that F has two subgroups say H, K such that $|H| = 10$ and $|K| = 6$. [so we can conclude that A_5 has two subgroups : one of order 10 and the other of order 6, since we know that A_5 is simple].

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com